The Mean And Variance Of The Exponential Of A Normally-Distributed Random Variate

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We are given the equation for the expected return on a stock from time zero to time t. The equation for expected return at time t is...

$$r_t = \mu t + \sigma \sqrt{t} z \tag{1}$$

In equation (1) above μ is the expected (mean) periodic return, σ is the periodic return standard deviation, t is the period number and z is a normally distributed random variate with mean zero and variance one. Stock return variable r_t is therefore a normally-distributed random variate with mean μt and variance $\sigma^2 t$.

We are also given the equation for stock price at time t. The equation for stock price at time t is...

$$S_t = S_0 e^{r_t} \tag{2}$$

We are tasked with determining the mean and variance of stock price at time t given equation (1) and equation (2) above.

Note: The distribution of stock price will be lognormal and the mean and variance will be that of a lognormal distribution.

The First Moment of the Distribution of Stock Price

The first moment of the distribution of S_t is...

$$\begin{aligned} \mathbb{E}\Big[S_t\Big] &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} S_t \delta z \\ &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} S_0 e^{r_t} \delta z \\ &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} S_0 e^{\mu t} e^{\sigma \sqrt{t} z} \delta z \\ &= S_0 e^{\mu t} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 + \sigma \sqrt{t} z} \delta z \end{aligned} \tag{3}$$

We will define the variable theta as...

$$\theta = z - \sigma \sqrt{t} \tag{4}$$

such that...

$$-\frac{1}{2}\theta^2 = -\frac{1}{2}z^2 + \sigma \sqrt{t} - \frac{1}{2}\sigma^2 t \tag{5}$$

We can now rewrite equation (3) above as...

$$
\begin{aligned}
\mathbb{E}\Big[S_t\Big] &= S_0 e^{\mu t} \int\limits_{-\infty-\sigma\sqrt{t}}^{\infty-\sigma\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2} e^{\frac{1}{2}\sigma^2 t} \delta\theta \\
&= S_0 e^{\mu t} e^{\frac{1}{2}\sigma^2 t} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2} \delta\theta \\
&= S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}
\end{aligned} \tag{6}
$$

Note that the last definite integral in equation (6) above is the cumulative normal distribution and integrates to one.

The Second Moment of the Distribution of Stock Price

The second moment of the distribution of S_t is...

$$
\begin{aligned}
\mathbb{E}\Big[S_t^2\Big] &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} S_t^2 \delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \Big[S_0 e^{r_t}\Big]^2 \delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} S_0^2 e^{2r_t} \delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} S_0^2 e^{2\mu t} e^{2\sigma\sqrt{t}z} \delta z \\
&= S_0^2 e^{2\mu t} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{2\sigma\sqrt{t}z} \delta z
\end{aligned} \tag{7}
$$

We will define the variable theta as...

$$
\theta = z - 2\sigma\sqrt{t} \tag{8}
$$

such that...

$$
-\frac{1}{2}\theta^2 = -\frac{1}{2}z^2 + 2\sigma\sqrt{t} - 2\sigma^2 t \tag{9}
$$

We can now rewrite equation (7) above as...

$$
\begin{aligned}
\mathbb{E}\Big[S_t^2\Big] &= S_0^2 e^{2\mu t} \int\limits_{-\infty-2\sigma\sqrt{t}}^{\infty-2\sigma\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2} e^{2\sigma^2 t} \delta\theta \\
&= S_0^2 e^{2\mu t} e^{2\sigma^2 t} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2} \delta\theta \\
&= S_0^2 e^{2\mu t + 2\sigma^2 t}
\end{aligned} \tag{10}
$$

Note that the last definite integral in equation (10) above is the cumulative normal distribution and integrates to one.

A Hypothetical Case

Current stock price is \$10.00 per share. The annual return on the stock is expected to be 15% with a standard deviation of 25%. What is the stock price mean and variance at the end of year 3?

Stock price mean is...

$$
\begin{aligned}
mean &= \mathbb{E}\Bigg[S_t\Bigg] \\
&= (10)exp((0.15)(3) + (0.5)(0.25^2)(3)) \\
&= 17.22
\end{aligned}
\tag{11}
$$

Stock price variance is...

$$
\begin{aligned}
variance &= \mathbb{E}\Bigg[S_t^2\Bigg] - \Bigg[\mathbb{E}\Bigg[S_t\Bigg]\Bigg]^2 \\
&= (10^2)exp((2)(0.15)(3) + (2)(0.25^2)(3)) \\
&= 357.87 - 17.22^2 \\
&= 61.34
\end{aligned}
\tag{12}
$$